Lecture Notes from Classroom Presentation

Noether’s Theorem in Classical Mechanics

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1. Introduction and Inspirations

The aim of learning a second course on classical mechanics is, on one hand, to know about certain mathematical techniques which make problem-solving easier. On the other hand, perhaps this is where a student first encounters and appreciates some general physical principles leading to the description of the dynamics of a system. These concepts are needed to the highest level of sophistication in theoretical Physics. For example, here one learns about the extremization of the action integral

\[ S = \int_{t_1}^{t_2} L \, dt \]  

which is found to be ubiquitously associated with almost all known physical theories. A similarly important concept is the importance of symmetries in a dynamical system, which will be of our interest here.

Let us begin with a simple observation at first. In your first semester, you learned about ‘inertial frames’- coordinate systems where the first law of Newton holds good. Subsequently, you also learned the transformations relating one such frame to another, namely, the ‘Galilean Transformations’. In the most general form, these consist of:-

1. a shift of origin by a constant vector \( \vec{r}_0 \),
   \[ \vec{r}' = \vec{r} + \vec{r}_0 \]  

2. a uniform motion with constant velocity \( \vec{v} \),
   \[ \vec{r}' = \vec{r} + \vec{v}t \]  

3. a constant rotation keeping the origin fixed,
   \[ \vec{r}' = R\vec{r} \]  

where \( R \) is an orthogonal \( 3 \times 3 \) rotation matrix, and finally,

4. a fixed change in time origin,
   \[ t' = t + t_0. \]  

So that, the complete general transformation can be represented in the form

\[ \vec{r}' = R\vec{r} + \vec{v}t + \vec{r}_0, t' = t + t_0. \]  

Clearly, to describe these transformation adequately, one needs 10 parameters:- 3 components for each of \( \vec{r}_0 \) and \( \vec{v} \), 3 more for the matrix \( R \) (e.g. the two polar angles specifying the orientation of the axis of rotation and the angle of rotation; of the 9 elements of the matrix \( R \), the conditions that it must be real, orthogonal and that it should preserve the length of a vector etc. leave room for only three parameters to construct its elements) and lastly, the value of \( t_0 \). Thus we need ten parameters to define the Galilean transformation. Surprisingly, number ten turns up in another seemingly unrelated scenario also. It is our common system of isolated particles.
Consider an n-particle system, free from external forces and interacting among themselves, say, gravitationally. The particles have masses \( m_i \), positions \( \vec{r}_i \), momenta \( p_i \) etc. 

For such a system, at the first glance, one can write the following conserved quantities:-

1. total linear momentum \( \vec{P} = \sum_{i=1}^{n} m_i \vec{v}_i \)
2. total angular momentum \( \vec{L} = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{v}_i \)
3. center-of-mass coordinates \( \vec{r}_{CM} = \sum_{i=1}^{n} \frac{m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} \) and 
4. total energy \( E = \sum_{i=1}^{n} \frac{\vec{p}_i^2}{2m_i} + U(\vec{r}_1, \vec{r}_2, ..., \vec{r}_n) \).

These conservation laws follow trivially from Newton’s laws, and were derived in the first semester explicitly. Here, if you notice, you will instantly see that there are exactly ten number of conserved quantities here too- three components for each of \( \vec{P}, \vec{L} \) and \( \vec{r}_{CM} \) and the total energy \( E \). This means that, there are as many constants of motion in an isolated dynamical system, as there are parameters in a general Galilean transformation. As these transformations leave an inertial system as inertial, they leave the form of equations of motion the same. That is, if we apply such transformations to our n-particle system center-of-mass coordinate,

\[
\ddot{\vec{r}}_{CM}(t) = 0 \iff \ddot{\vec{r}}_{CM}(t') = 0.
\]

Thus, it is natural to ask whether there is any one-to-one relation between the invariance of a physical system under a set of transformations and its integrals of motion (i.e., conserved quantities). This question is at the heart of Emmy Noether’s theorem, which proves there is really such a correspondence under certain circumstances. In later part of this lecture, we’ll discuss her theorem in the Lagrangian framework of classical mechanics in two equivalent ways. But, at first, let us look at some of the simplest consequences of invariance of an equation under certain transformation.

### 2. Fundamental Consequences of Symmetry

Consider the following examples. In what follows, we shall appreciate some of the very first applications of symmetry, i.e., of the transformations which conserve the form of an equation.

**Example 2.1**

An well-known algebraic equation is

\[ z^4 - 1 = 0, \]  

where \( z \) is a complex number. If you are asked to find its roots, you will probably factorize the polynomial and find them. But my approach is different. One of the roots is obviously
1, but what about the others? Consider a variable transformation from $z$ to $z'$ defined as

$$z' = iz.$$  \hfill (e2.1.2)

In the terms of this new variable, equation (e2.1.1) takes the form

$$\left( \frac{z'}{i} \right)^4 - 1 = 0$$

or,

$$z'^4 - 1 = 0,$$  \hfill (e2.1.3)

which is same as equation (e2.1.1) in its look, albeit a change in the dummy variable. So, as $z = 1$ was a solution of equation (e2.1.1), hence $z' = 1$ is a solution of equation (e2.1.3). Next, we see that the equations (e2.1.1) and (e2.1.3) are basically the same equation. So, $z' = 1$ or correspondingly from equation (e2.1.2), $z = \frac{1}{i} = -i$ should also be a root of our original equation (e2.1.1). Thus, we have obtained a solution just by exploiting symmetry properties of the equation. In your school days, many of you were much bothered about factorizing lengthy algebraic expressions using various tricks. This method bypasses all those things.

**Exercise 1**

Obtain the other roots of equation (e2.1.1) by using the symmetry transformations

$$z'' = -z$$

and

$$z''' = -iz.$$  

**Exercise 2**

Plot the solutions of equation (e2.1.1) in a Re($z$)-Im($z$) plot and discuss how the above transformations relate the solutions among themselves geometrically.

These ideas are useful for differential equations as well. Here, the transformations usually depend on one or more parameters which can be varied continuously. So, they define a continuous set of transformations. Let us see an example.

**Example 2.2**

Consider the familiar differential equation

$$\frac{d^2 y}{dx^2} + y = 0.$$  \hfill (e2.2.1)

We know a solution $y = \cos x$. But the fact that the transformations defined by

$$y' = \frac{y}{A}$$  \hfill (e2.2.2)
with $A$ being a nonzero number and

$$x' = x - \phi$$

(e2.2.3)

leave the form of the equation (e2.2.1) the same, leads us to the most general family of solutions

$$y = A\cos(x + \phi).$$

(e2.2.4)

Here the parameters of the transformations are $A$ and $\phi$ which can be changed smoothly. These transformations have interesting geometric interpretation. In the phase space of the differential equation(e2.2.1), the curves described by the general solutions belong to a family of circles of different radii. The transformations defined by the equation (e2.2.2) switches the circles among themselves by changing their radii through the change in $A$, while those of the equation (e2.2.3) carry a point along a particular circle.

**Exercise 3**

This is an exploration rather than an exercise, at least at your level. Consider your favorite Hamiltonian system(s) and write down the Hamilton’s equations of motion. Next find the set of transformations which leave them covariant. Try to examine what are their general characteristics.

By this time, looking at the examples above and solving the exercises, you should be pretty sure of the fact that if a set of transformations conserve the form of an equation, then it inter-converts the solutions of the equation. We will use this fact later to the Euler-Lagrange equation of classical mechanics. Before that, let us quickly recapitulate the theoretical framework of classical mechanics at first.

3. The Lagrangian Formalism and Principle of Stationary Action

In the formal language of analytical mechanics, a system with $f$ degrees of freedom is described by a set of $f$ generalized coordinates in the configuration space, which are denoted by $[q_i], i = 1, 2, ..., f$ and the Lagrangian $L = L(q, \dot{q}, t)$. The canonically conjugate momenta corresponding to the co-ordinates are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

(3.1)

and with these, one defines the Hamiltonian as

$$H(q, p, t) = \sum_{i=1}^{f} p_i \dot{q}_i(q, p, t) - L(q, \dot{q}, q, p, t, t).$$

(3.2)
With equation (3.2), the action integral in equation (1.1) becomes

$$S = \int_{t_1}^{t_2} \left[ \sum_{i=1}^{f} p_i \dot{q}_i(q, p, t) - H(q, p, t) \right] dt. \tag{3.3}$$

Now, of all the curves in the configuration space between the time interval \( t_1 \) to \( t_2 \), the actual physical trajectory \( q = q(t) \) taken by the system is that which extremizes the action integral \( S \) of equation (3.3). That is, for first-order variations about the real path, the action integral is stationary, i.e.,

$$\delta S = 0, \tag{3.4}$$

provided that the endpoints \( q(t_1) \) and \( q(t_2) \) of the curves are kept fixed. On the other hand, if we vary \( q_i(t) \) and/or time \( t \) at the endpoints also, then the only contribution to \( \delta S \) comes from the endpoint variations, equal to, say,

$$\delta S = G(t_2) - G(t_1) \tag{3.5}$$

with the explicit form of \( G \) depending on the system of our interest and the form of the Lagrangian taken. Let us see what it may look like. The left hand side of equation (3.4) maybe written as

$$\delta S = \delta \int_{t_1}^{t_2} \left[ \sum_{i=1}^{f} p_i \frac{dq_i}{dt} - H \right] dt$$

$$= \delta \int_{t_1}^{t_2} \left[ \sum_{i=1}^{f} p_i dq_i - H dt \right]$$

$$= \int_{t_1}^{t_2} \left[ \sum_{i=1}^{f} \delta p_i dq_i + \sum_{i=1}^{f} p_i \delta(q_i) - \delta H dt - H \delta(dt) \right]$$

(as \( \delta \) and \( d \) are independent operators, so we take \( \delta \) into the integrand and apply product rule)

$$= \int_{t_1}^{t_2} \left[ \sum_{i=1}^{f} \dot{q}_i \delta p_i + \sum_{i=1}^{f} p_i \frac{d(\delta q_i)}{dt} - \delta H + \frac{dH}{dt} \delta(dt) \right] dt \tag{3.6}$$

as \( d \) and \( \delta \) operators commute. Now \( H = H(q, p, t) \) which enables us to write, using chain rule,

$$\delta H = \sum_{i=1}^{f} \left[ \frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i \right] + \frac{\partial H}{\partial t} \delta t. \tag{3.7}$$

Using equations (3.6) and (3.7) we can write further

$$\delta S = \int_{t_1}^{t_2} \left[ \sum_{i=1}^{f} (\dot{q}_i - \frac{\partial H}{\partial p_i}) \delta p_i - \sum_{i=1}^{f} (\dot{p}_i + \frac{\partial H}{\partial q_i}) \delta q_i + (\frac{dH}{dt} - \frac{\partial H}{\partial t}) \delta t \right] dt + \int_{t_1}^{t_2} d \left[ \sum_{i=1}^{f} p_i \delta q_i - H \delta t \right] \tag{3.8}$$

where we have added and subtracted the terms \( \sum_{i=1}^{f} \dot{p}_i \delta q_i \) and \( H \delta t \) and assembled the terms according to the product rule to obtain the total differential in the last term. Now the
comparison of the equations (3.5) and (3.8) reveals many important things. The last term, after integration will yield the boundary terms. It would vanish if we do not permit boundary variations, i.e., for $t = t_{1,2}$, $\delta q = 0$ and $\delta t = 0$ at the boundary. The other terms will add up to zero, by the principle of stationary action. For the reason that $\delta q$, $\delta p$ and $\delta t$ are independent variations, we conclude that the coefficients of each of them in equation (3.0) should vanish identically. From the coefficients of $\delta p_i$ and $\delta q_i$, we get respectively,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (3.9)$$

and

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (3.10)$$

for all $i = 1, 2, ..., f$. We instantly recognize them as the well-known Hamilton’s equations of motion. The boundary term is given by

$$G = \sum_{i=1}^{f} p_i \delta q_i - H \delta t \quad (3.11).$$

In the next section, this boundary term will provide a key to understand the relation between symmetry and conservation principle and finally lead us to the Noether’s theorem. Once we have obtained the equations of motion from the action principle in this section, let us now concentrate on the boundary terms.

4. Boundary Contributions and the Intuition into Noether’s Theorem

In the previous section, we have seen that for variations about the physical trajectory in configuration space, the action is stationary except the contributions at the boundary. Now, if for certain particular variations, the change in action vanishes, at least in the first-order, then we get an interesting situation. In those cases, the action is invariant under those variational transformations. And in those cases, even allowing for endpoint variations,

$$\delta S = G(t_2) - G(t_1) = 0 \quad (4.1)$$

between time $t_1$ and $t_2$. Now, Our choice of endpoints can be arbitrary and this relation (4.1) is true for all of them. We could have kept one of the endpoints, say, $t_1$ fixed and taken a new $t_2$ along the path and this relation would hold for that time as well. This can happen if and only if $G$ is a constant along the path, i.e., the boundary contribution to the variation in action is a conserved quantity of motion. This is how invariance and conservation are directly related in a dynamical system. This is the Noether’s theorem. Before giving its standard statement, let us see some of its consequences at first. In the examples below, all the variations of path are very small, such that only the first-order contributions to the action are necessary to be taken into the account.
Example 4.1
Suppose that the action of our system is invariant under a fixed translation in space, i.e., $\delta S = 0$ under the variation $\delta q_i = \epsilon_i = \text{constant}$, $\delta t(t_{1,2}) = 0$. Then, equation (3.11) gives us $G = \vec{p} \cdot \vec{c} = \text{constant}$. As $\vec{c}$ can be arbitrary, so $\vec{p}$ is a constant of motion, i.e., momentum is conserved. So, invariance under space translation implies momentum conservation.

Example 4.2
Suppose now our action is invariant under a fixed time translation $\delta t = \epsilon = \text{constant}$, $\delta q_i(t_{1,2}) = 0$. If this be the situation, then, equation (3.11) yields $H = \text{constant}$, i.e., time translation symmetry implies energy conservation.

The third familiar implication is left as an exercise.

Exercise 4
Consider a Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(\vec{r})$$

where the potential is radially symmetric and hence no particular orientation in space is preferred. Write down the action integral and show explicitly that it is invariant under the fixed rotation $\delta x_i = \epsilon_{ijk}\delta \omega_j x_k$, $\delta t(t_{1,2}) = 0$, i.e., $\vec{\delta r} = \vec{\omega} \times \vec{r}$, where $\vec{\omega} = \text{constant}$. Finally, find the expression for $G$ and from it, show that the angular momentum is conserved. This is one instance of the fact that angular momentum conservation follows from the invariance under rotation.

Apart from the above three familiar examples, there are some other non-trivial examples of conservation, applicable to only special systems. Often, in those situations, the variations $\delta q$ and $\delta p$ may not be arbitrary or trivial. One such example is the conservation of the famous Laplace-Runge-lenz vector $\vec{A} = \frac{L \times \vec{p}}{mk} - \frac{\vec{r}}{r}$ in the Kepler problem. The vector has an interesting history and from dynamical aspects, it is important for many reasons. Firstly, the existence of closed orbits in Kepler potential is attributed to it. Secondly, and more marvelously, this symmetry is the symmetry of a hypersphere in four-dimensional space. Not just that, we know that some dynamical aspects of the quantum mechanical hydrogen atom can be studied with intuitions from the Kepler problem. It was in 1935 that the famous physicist Vladimir Fock discovered that the electron in a one-electron atom moves as though it were in an environment with the symmetry of a hypersphere in four-space. In that case, the conservation of the above vector is manifested in the energy degeneracy of the principal quantum number. Apart from these problems, similar looking conserved quantities turn up in Hooke-type potential also. An interested advanced reader is invited to explore them in the references given below.
So far, we have derived and discussed Noether’s theorem intuitively starting from the action principle. There is however, an alternative approach using the Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}
\]

for \( i = 1, 2, \ldots, f \). This version is mathematically more simple, though it may seem somewhat less intuitive than what we have discussed. But it is the more familiar version of the theorem commonly found in the standard textbooks. Let us discuss about it.

5. Noether’s Theorem from the Euler-Lagrange Equation

Let us state the theorem at first.

**Theorem (Noether)**

In an autonomous system, if the Lagrangian \( L(q, \dot{q}) \) is invariant under the transformation

\[
q \rightarrow q'(s, q),
\]

where \( s \) is a real continuous parameter and \( s = 0 \) corresponds to identity transformation, i.e., \( q'(0, q) = q \), then, there exists a conserved quantity of motion given by

\[
I(q, \dot{q}) = \sum_{i=1}^{f} \left[ \frac{\partial L}{\partial \dot{q}_i} \frac{d}{ds} q'_i(s, q) \right]_{s=0}.
\]  

(5.1)

The equivalence of the theorem with the ideas we discussed previously is not difficult to understand once we know that it is none but the Euler-Lagrange equations which extremizes the action. Rather than studying about the variation of the action itself, here we use the Euler-Lagrange equations to prove it. We will also use the ideas we have got in section 2.

**Proof** From section 2, we know if a transformation leaves a differential equation invariant, then it switches its solutions. So if \( q = f(t) \) is a solution of the equation (4.2), then, by assumption in the theory, \( q' = F(s, t) \) is also a solution of the same equation. Furthermore, due to the invariance of the Lagrangian itself under the transformation, we are able to write

\[
\frac{d}{ds} L(F(s, t), \dot{F}(s, t)) = \sum_{i=1}^{f} \left[ \frac{\partial L}{\partial F_i} \frac{dF_i}{ds} + \frac{\partial L}{\partial \dot{F}_i} \frac{d\dot{F}_i}{ds} \right] = 0.
\]

(5.2)

With these, to show the conservation of \( I \), we take its time derivative and show that it is zero.

\[
\frac{dI}{dt} = \frac{d}{dt} \left[ \sum_{i=1}^{f} \frac{\partial L}{\partial q_i} \frac{d}{ds} q'_i(s, q) \right]_{s=0}
\]
\[
= \sum_{i=1}^{f} \left\{ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \frac{dq_i}{ds} \bigg|_{s=0} + \frac{\partial L}{\partial q_i} \frac{d}{dt} \left( \frac{dq_i}{ds} \bigg|_{s=0} \right) \right\}
\]
(by product rule)
\[
= \sum_{i=1}^{f} \left\{ \frac{\partial L}{\partial q_i} \frac{dq_i}{ds} \bigg|_{s=0} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \left( \frac{dq_i}{ds} \bigg|_{s=0} \right) \right\}
\]
(using the Euler-Lagrange equation)
\[
= \sum_{i=1}^{f} \left( \frac{\partial L}{\partial F_i} \frac{dF_i}{ds} + \frac{\partial L}{\partial \dot{F}_i} \frac{d\dot{F}_i}{ds} \bigg|_{s=0} \right)
\]
(we use the fact that \(s=0\) corresponds to \(F_i = q_i\) and then the total quantity needs to be evaluated at \(s = 0\). We also use \(\frac{d}{dt} \frac{dF}{ds} = \frac{d\dot{F}}{ds}\))
\[
= \frac{dL}{ds} \bigg|_{s=0}
\]
(by product rule we assemble the terms)
\[
= 0.
\]
(by equation 5.2) So, as the total time derivative of \(I\) equal to zero, it is a conserved quantity.

Here also we get the familiar conservation laws like before using our translational and rotational transformations. Linear and angular momentum conservation follows trivially and left for you as an exercise.

**Exercise 6**

Considering the one-parameter transformation \(\vec{r}' = \vec{r} + s\hat{i}\), show that if the Lagrangian is invariant under translation along a direction, then \(I\) is the projection of the total linear momentum on that direction and hence is conserved.

**Exercise 7**

Write down the transformation laws denoting rotation about the \(z\)-axis. Show that the invariance of the Lagrangian under such transformation implies that the \(z\)-component of the total angular momentum is constant.

The energy conservation from time-translational symmetry is somewhat less trivial. It requires consideration of time as a coordinate-like variable through parametrization of both generalized coordinates and time. Then the symmetry of time translation gives the conserved quantity \(I\) in equation (5.1) as the Hamiltonian. As we noted in the last section, there are
generalizations of the theorem considering more general transformations. Also, the fact that the Lagrangian can be gauge-transformed is also of help here. Such generalizations usually contain too lengthy calculations and are omitted here. An interested student is asked to refer to textbooks like “Mechanics” by Florian Scheck (4-th edition).

6. References and Discussions

The material discussed above along with the exercises is meant to be sufficient for a first learner. However, for those who wish to read more about the topic, here are my suggestions. These are also the materials I consulted to prepare this note.

The most common version of Noether’s theorem is that discussed in section 5. Almost all standard advanced undergraduate or graduate text of classical mechanics contains those. The reader can consult “Classical Mechanics” by Herbert Goldstein or “Mechanics” by Florian Scheck. The latter generalizes the theorem by exploiting the gauge transformations of the Lagrangian and allowing for more general transformations. However, the intuition from action principle is specially adapted from Richard Feynmann’s “The Character of Physical Law” and its mathematical machinery is developed in “Classical and Quantum Dynamics” by Walter Dittrich and Martin Reuter. Both Goldstein and they have discussion on the Laplace-Runge-lenz vector. For general friendly discussions and illustrations on implications of symmetry in dynamical systems is given in “Dynamical Symmetry” by Carl Wulfman.

Finally, I end with the saying that you have learned something which is useful in not only classical mechanics, but in every branch of theoretical Physics. In every theory, symmetry and conservation laws play a key role. The intuitions and methods we have developed so far are generalizable for other forms of actions and interactions, with more mathematically rigorous framework. This is just the beginning.